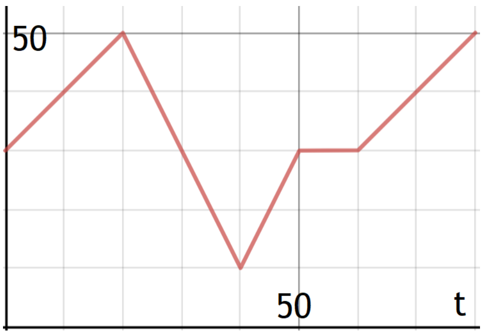
MATH 102:112, CLASS 13 SOLUTIONS (THURSDAY OCT 18)

(1) Calculate the derivatives.

$\frac{1}{3x^3+2}$
Answer: $-\frac{9x^2}{(3x^3+2)^2}$

$x\sqrt{4x^2-1}$
Answer: $\sqrt{4x^2-1}+\frac{4x^2}{\sqrt{4x^2-1}}$

(2) The level of pollution in a lake is dependent on the population of humans by the lake. Let $P(H) = H^2$ equal the amount of human-created pollution, where H is the number of humans (in thousands). Regular census-taking yields the graph of $y = H(t)$ shown. (t in years)



We would like to understand how the pollution levels change with time.

(a) Calculate $\frac{dP}{dt}$ at $t = 30$.

Solution: $\frac{dP}{dt} = \frac{dP}{dH} \cdot \frac{dH}{dt} = 2H(30)H'(30) = 2(30)(-2) = -120$

(b) Calculate $\frac{dP}{dt}$ at $t = 10$.

Solution: $\frac{dP}{dt} = 2H(10)H'(10) = 2(40)(1) = 80$

(c) Calculate $\frac{dP}{dt}$ at $t = 55$.

Solution: $\frac{dP}{dt} = 2H(55)H'(55) = 2(30)(0) = 0$

(3) A circular bacterial colony has radius $r(t)$ and area $A(t)$ (so that $A(t) = \pi(r(t))^2$).

(a) Suppose that $r(t)$ grows at a constant rate of 2. Calculate $\frac{dA}{dt}$ when $r = 5$.

Solution: The following equation relates the radius and the area.

$$A = \pi r^2$$

If we take the derivative of both sides with respect to t, we get $A'(t) = 2\pi r(t)r'(t)$, or written another way,

$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$$

We are told that $\frac{dr}{dt} = 2$ and $r = 5$. Then $\frac{dA}{dt} = 2\pi(5)(2) = 20\pi$.

(b) Suppose instead that $A(t)$ grows at a constant rate of 20π. Calculate $\frac{dr}{dt}$ when $r = 5$.

Solution: We can use the same equation $\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$. We are given that $\frac{dA}{dt} = 20\pi$ and $r = 5$, and we want to calculate $\frac{dr}{dt}$. Therefore, $20\pi = 2\pi(5)\frac{dr}{dt}$ which means $\frac{dr}{dt} = 2$.

(4) (More optimization practice) An animal is deciding what proportion of its food-gathering time, x, it should allot between two different types of food (where $0 \leq x \leq 1$).

(a) Suppose there are two types of food, 1 and 2, and the nutrition gained from spending x portion of time on each is $F_1(x) = x^{1/2}$ and $F_2(x) = Nx$ for some positive constant N. What is the maximum amount of nutrition the animal can gain, and for what value of x does this happen? Your answer will depend on N.

Solution: If the animal spends x of its time on the first type of food, and $1-x$ on the second, then its total nutritional gain is $F(x) = x^{1/2} + N(1-x)$. Find the critical points:

$$F'(x) = \frac{1}{2\sqrt{x}} - N = 0 \implies \sqrt{x} = \frac{1}{2N} \implies x = \frac{1}{4N^2}$$

- If $N < 1/2$, then this critical points does not lie in the interval $[0, 1]$. We therefore can find the maximum by testing the endpoints:

$$F(0) = 0^{1/2} + N(1-0) = N \qquad F(1) = 1^{1/2} + N(1-1) = 1$$

  and find that the maximum occurs at $x = 1$, because $N < 1/2$
- If $N \geq 1/2$, then the critical point above **does** lie in the interval $[0, 1]$. To determine if this is a maximum, we can either plug back into $F(x)$, or calculate the second derivative. We will do the second option.

$$F'(x) = \frac{1}{2}x^{1/2} - N \implies F''(x) = -\frac{1}{4}x^{-3/2}$$

  It is clear that $F''(\frac{1}{4N^2})$ will be negative, thus $x = \frac{1}{4N^2}$ is a **local maximum**. It follows that it is a global maximum.

(b) Same question, but for $F_1(x) = x^2$ and $F_2(x) = Nx$.

Solution: Now, $F(x) = x^2 + N(1-x)$. Find the critical points:

$$F'(x) = 2x - N = 0 \implies x = N/2$$

- If $N > 2$, then the critical point does not lie in the interval $[0, 1]$. We can then plug in

$$F(0) = 0^2 + N(1-0) = N \qquad F(1) = 1^2 + N(1-1) = 1$$

  and find that the maximum occurs at $x = 0$.
- If $N \leq 2$, then the critical point **does** lie in the interval $[0, 1]$. We'll use the second derivative test as before

$$F''(x) = 2 \implies F''(N) = 2$$

  and so this critical point is a local minimum! So the global maximum still occurs at $x = 0$.